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The Generalized Riemann, Simple, Dominated and Improper Integrals

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1. INTRODUCTION

A powerful generalization of the Riemann integral has been introduced by making an innocent-looking modification in the usual definition. This *generalized Riemann integral* was defined in 1957 by Kurzweil [6]. It was independently defined and extensively studied and generalized by Henstock [3–5] who called it the Riemann-complete integral. Although this integral has been popularized somewhat (cf. [7]), it still is not as well known as it deserves to be. Among the virtues of this powerful integral are the following:

(1) Every Lebesgue integrable function is generalized Riemann integrable and the values of the integrals are the same. The same holds for functions integrable in several other senses.

(2) The monotone and dominated convergence theorems of the Lebesgue theory can be stated so as to hold for the generalized Riemann integral.

(3) Improper Riemann integrals are "proper" generalized Riemann integrals.

(4) If f' exists throughout $[a, b]$, then it is generalized Riemann integrable on $[a, b]$ and the value of the integral is $f(b) - f(a)$.

Unaware of the generalized Riemann integral, Haber and Shisha [1, 2] defined and studied the "simple integral" and Osgood and Shisha [10, 11] defined and studied the "dominated integral". The aims in forming these concepts were (i) to replace the improper Riemann integrals, which are iterated limits, by single limits and (ii) to allow a wide use of standard quadrature formulas in the evaluation of singular integrals. In Sections 2 and 3 we shall show how the simple and dominated integrals fit into the framework of the generalized Riemann integral in a natural and very simple way. In Section 4 analogous results are obtained for the improper Riemann integrals.

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2. Definition of the Generalized Riemann Integral; Connection with the Dominated Integral

Definition 1 (cf. [5, p. 82]). A real function f is generalized Riemann integrable on $[a, b]$ $(-\infty < a < b < \infty)$ iff it is defined there and there is a real number I with the following property: For each $\varepsilon > 0$ there is a positive function $\delta_\varepsilon(t)$ on $[a, b]$ such that $|I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})| < \varepsilon$ whenever $a = x_0 < \cdots < x_n = b$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1,..., n$. If such an I exists it is unique and is called the generalized Riemann integral of f on $[a, b]$.

Notice that if, for every $\varepsilon > 0$, $\delta_\varepsilon(t)$ is a constant c_ε, then generalized Riemann integrability on $[a, b]$ is precisely (proper) Riemann integrability there. Suppose $a = 0$, $b = 1$. We show that when $\delta_\varepsilon(t)$ is retricted to be a linear function: $\delta_\varepsilon(t) = c(\varepsilon)t$, $0 < t \leqslant 1$, then generalized Riemann integrability is equivalent to dominant integrability. First, we state the definition of the dominated integral; this was formulated with a function unbounded near zero in mind.

Definition 2 [10]. A real function f on $(0, 1]$ is dominantly integrable iff there is a real number I with the following property:

For each $\varepsilon > 0$ there are numbers $\Delta(\varepsilon)$ and $\psi(\varepsilon)$ with $0 < \Delta(\varepsilon) < 1$, $0 < \psi(\varepsilon) < 1$ such that

$$\left| I - \sum_{k=2}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon \tag{1}$$

whenever $0 < x_1 < \cdots < x_n = 1$, $x_1 < \psi(\varepsilon)$, and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_{k-1}/x_k > 1 - \Delta(\varepsilon)$ for $k = 2,..., n$.

If there is such an I it is unique and is called the dominated integral of f.

Theorem 1. *Let f be a real function on $(0, 1]$ and define $f(0) = 0$. Then f is dominantly integrable iff there is a real number I with the following property:*

For each $\varepsilon > 0$ there are positive numbers $c_1(\varepsilon)$ and $c_2(\varepsilon)$ such that, defining

$$\begin{aligned} \delta_\varepsilon(t) &= c_1(\varepsilon)t, && \text{if} \quad 0 < t \leqslant 1, \\ &= c_2(\varepsilon), && \text{if} \quad t = 0, \end{aligned} \tag{2}$$

we have

$$\left. \begin{aligned} &\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon \\ &\textit{whenever} \quad 0 = x_0 < x_1 < \cdots < x_n = 1 \quad \textit{and} \quad x_{k-1} \leqslant t_k \leqslant x_k, \\ &x_k - x_{k-1} < \delta_\varepsilon(t_k) \textit{ for } k = 1,\ldots, n. \end{aligned} \right\} \tag{2}$$

Proof. ($\Rightarrow$) Assume (1) and define for every $\varepsilon > 0$,

$$\begin{aligned} \delta_\varepsilon(t) &= \Delta(\varepsilon)t, \qquad \text{if} \quad 0 < t \leqslant 1, \\ &= \psi(\varepsilon), \qquad \text{if} \quad t = 0. \end{aligned}$$

Given $\varepsilon > 0$, let $0 = x_0 < x_1 < \cdots < x_n = 1$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1,\ldots, n$. Then for all $2 \leqslant k \leqslant n$, $x_k - x_{k-1} < \Delta(\varepsilon)\, t_k \leqslant \Delta(\varepsilon)\, x_k$ and hence $x_{k-1}/x_k > 1 - \Delta(\varepsilon)$. Also $\delta_\varepsilon(t_1) > x_1 - x_0 = x_1 \geqslant t_1 \geqslant \Delta(\varepsilon)\, t_1$ and so $t_1 = 0$ and $\delta_\varepsilon(t_1) = \psi(\varepsilon)$. Thus $x_1 = x_1 - x_0 < \psi(\varepsilon)$ and by (1),

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| = \left| I - \sum_{k=2}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon.$$

($\Leftarrow$) Assume (2) and define for every $\varepsilon > 0$, $\psi(\varepsilon) = \min\{c_2(\varepsilon), \frac{1}{2}\}$, $\Delta(\varepsilon) = c_1(\varepsilon)/(1 + c_1(\varepsilon))$ so that $0 < \psi(\varepsilon) < 1$, $0 < \Delta(\varepsilon) < 1$. Given $\varepsilon > 0$, let $0 < x_1 < \cdots < x_n = 1$, $x_1 < \psi(\varepsilon)$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_{k-1}/x_k > 1 - \Delta(\varepsilon)$ for $k = 2,\ldots, n$. Defining $x_0 = 0$ and $t_1 = 0$, we have $x_1 - x_0 < \psi(\varepsilon) \leqslant c_2(\varepsilon) = \delta_\varepsilon(t_1)$. Also if $2 \leqslant k \leqslant n$, then $x_{k-1} > (1 - \Delta(\varepsilon))\, x_k$ and hence

$$x_k - x_{k-1} < \Delta(\varepsilon)\, x_k < \frac{\Delta(\varepsilon)}{1 - \Delta(\varepsilon)}\, x_{k-1} \leqslant \frac{\Delta(\varepsilon)}{1 - \Delta(\varepsilon)}\, t_k = c_1(\varepsilon)\, t_k = \delta_\varepsilon(t_k).$$

Hence by (2), $|I - \sum_{k=2}^{n} f(t_k)(x_k - x_{k-1})| = |I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1})| < \varepsilon$ and so f is dominantly integrable. This completes the proof.

3. Connection with the Simple Integral

The simple integral was defined for real functions on $[0, \infty)$. For comparison, we now define the generalized Riemann integral on $[0, \infty)$ by modifying slightly the finite interval definition.

Definition 3 (cf. [5, p. 83]). A real function f is generalized Riemann integrable on $[0, \infty)$ iff it is defined there and there is a real number I with the following property:

For each $\varepsilon > 0$ there is a real positive function $\delta_\varepsilon(t)$ on $[0, \infty)$ and a positive number $B(\varepsilon)$ such that

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon$$

whenever $0 = x_0 < \cdots < x_n$, $x_n > B(\varepsilon)$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1, \ldots, n$.

When, for every $\varepsilon > 0$, $\delta_\varepsilon(t)$ is a constant c_ε, then generalized Riemann integrability on $[0, \infty)$ is precisely simple integrability defined in [1, 2]. Hence the relationship of simple integrability to generalized Riemann integrability on $[0, \infty)$ is the same as the relationship of (proper) Riemann integrability to generalized Riemann integrability on intervals $[a, b]$, $-\infty < a < b < \infty$. Thus from the point of view of the generalized Riemann integral, the simple integral rather than the Riemann improper integral seems to be the natural extension of the Riemann integral to $[0, \infty)$.

4. Connection with Improper Riemann Integrals

In this section we characterize the improperly Riemann integrable functions as those functions which are generalized Riemann integrable with a *monotone* function $\delta_\varepsilon(t)$ for every fixed $\varepsilon > 0$. As is customary we say a real function f defined on $(a, b]$, with $-\infty < a < b < \infty$, is *improperly Riemann integrable on* $(a, b]$ iff f is Riemann integrable on $[s, b]$ for each s with $a < s < b$, and $\lim_{s \to a+} \int_s^b f$ exists (finite), in which case we denote the limit $\int_a^b f$.

The following simple but fundamental result will be useful:

Henstock's Lemma (cf. [5, Theorem 5]). *Let a real function f be generalized Riemann integrable on $[a, b]$, $-\infty < a < b < \infty$, let $\varepsilon > 0$ be given and let $\delta_\varepsilon(t)$ be as in Definition 1. If $a = x_0 < \cdots < x_n = b$; $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1, 2, \ldots, n$, and N is a subset of $\{1, 2, \ldots, n\}$, then*

$$\left| \sum_{k \in N} \left(\int_{x_{k-1}}^{x_k} f \right) - f(t_k)(x_k - x_{k-1}) \right| \leqslant \varepsilon.$$

(For $k = 1, 2, \ldots, n$, $\int_{x_{k-1}}^{x_k} f$ denotes the generalized Riemann integral which necessarily exists for $k = 1, 2, \ldots, n$. An "empty sum" is 0.)

THEOREM 2. *Let f be a real function on $(a, b]$ with $-\infty < a < b < \infty$ and define $f(a) = 0$. Then f is improperly Riemann integrable on $(a, b]$ iff there is a real number I with the following property:*

> *For each $\varepsilon > 0$ there is a positive function $\delta_\varepsilon(t)$ on $[a, b]$ which is nondecreasing on $(a, b]$ and such that*
>
> $$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon \qquad (3)$$
>
> *whenever $a = x_0 < \cdots < x_n = b$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1,\ldots, n$.*

Proof of Theorem 2. ($\Rightarrow$) This half of the proof was suggested by an argument in [7, p. 66]. Let $\varepsilon > 0$ be given. Let $(c_m)_{m=0}^{\infty}$ be a strictly decreasing sequence with $c_0 = b$ and $\lim_{m\to\infty} c_m = a$. Since f is Riemann integrable on $[c_2, b]$, there is a number γ_1 such that $0 < \gamma_1 \leqslant \min(c_0 - c_1, c_1 - c_2)$ and

$$\left| \int_{c_2}^{c_0} f - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon/2^2$$

whenever $c_2 = x_0 < \cdots < x_n = c_0$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \gamma_1$ for $k = 1,\ldots, n$. For $m = 2, 3,\ldots$ let γ_m satisfy $0 < \gamma_m \leqslant c_m - c_{m+1}$, $\gamma_m \leqslant \gamma_{m-1}$, and

$$\left| \left(\int_{c_{m+1}}^{c_{m-2}} f \right) - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon/2^{m+1}$$

whenever $c_{m+1} = x_0 < \cdots < x_n = c_{m-2}$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \gamma_m$ for $k = 1,\ldots, n$.

Now define $\delta_\varepsilon(t) = \gamma_m$ if $c_m < t \leqslant c_{m-1}$, $m \geqslant 1$, so that $t - \delta_\varepsilon(t) \geqslant t - c_m + c_{m+1} > c_{m+1} > a$. Then $\delta_\varepsilon(t)$ is nondecreasing on $(a, b]$. Since $\lim_{s\to a+} \int_s^b f = \int_a^b f$, there is a number S with $a < S < b$ such that $|\int_a^b f - \int_s^b f| < \varepsilon/2$ whenever $a < s < S$. Define $\delta_\varepsilon(a) = S - a$.

Let $a = x_0 < \cdots < x_n = b$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1,\ldots, n$. If $t_1 > a$, then $x_1 - \delta_\varepsilon(t_1) \geqslant t_1 - \delta_\varepsilon(t_1) > a = x_0$, so $x_1 - x_0 > \delta_\varepsilon(t_1)$ which is false. Hence $t_1 = a$, $\delta_\varepsilon(t_1) = S - a$ and $x_1 < S$ (so $n \geqslant 2$).

We have

$$\left| \left(\int_a^b f \right) - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right|$$
$$\leqslant \left| \left(\int_a^b f \right) - \left(\int_{x_1}^b f \right) \right| + \left| \left(\int_{x_1}^b f \right) - \sum_{k=2}^{n} f(t_k)(x_k - x_{k-1}) \right|.$$

The first difference on the right-hand side is $<\varepsilon/2$. Consider the second difference. For $m = 1, 2, \ldots$ let N_m be the set of integers k for which $2 \leqslant k \leqslant n$ and $t_k \in (c_m, c_{m-1}]$. If $m \geqslant 1$ and $k \in N_m$, then

$$x_{k-1} \geqslant x_{k-1} - x_k + t_k > -\gamma_m + t_k > -\gamma_m + c_m \geqslant c_{m+1}.$$

Hence if $m \geqslant 2$ and $k \in N_m$, then (with $\gamma_0 = c_0 - c_1$)

$$c_{m+1} < x_{k-1} < x_k \leqslant x_k - x_{k-1} + t_k < \gamma_m + t_k$$
$$\leqslant \gamma_{m-2} + c_{m-1} \leqslant c_{m-2}$$

while if $k \in N_1$, then

$$c_2 < x_{k-1} < x_k \leqslant c_0.$$

Let $m \geqslant 1$. By Henstock's Lemma

$$\left| \sum_{k \in N_m} \left(\int_{x_{k-1}}^{x_k} f \right) - f(t_k)(x_k - x_{k-1}) \right| \leqslant \varepsilon/2^{m+1}.$$

Hence

$$\left| \left(\int_{x_1}^{b} f \right) - \sum_{k=2}^{n} f(t_k)(x_k - x_{k-1}) \right| = \left| \sum_{k=2}^{n} \left(\int_{x_{k-1}}^{x_k} f \right) - f(t_k)(x_k - x_{k-1}) \right|$$
$$= \left| \sum_{m=1}^{\infty} \sum_{k \in N_m} \left(\int_{x_{k-1}}^{x_k} f \right) - f(t_k)(x_k - x_{k-1}) \right| \leqslant \sum_{m=1}^{\infty} \varepsilon/2^{m+1} = \varepsilon/2.$$

Thus

$$\left| \left(\int_{a}^{b} f \right) - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon.$$

So (3) is satisfied with $I = \int_a^b f$.

($\Leftarrow$) Let $a < s < b$ and let $\varepsilon > 0$. If $s = x_0 < \cdots < x_n = b$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(s)$ for $k = 1, \ldots, n$, then we have $x_k - x_{k-1} < \delta_\varepsilon(t_k)$, $k = 1, 2, \ldots, n$, since $\delta_\varepsilon(t)$ is nondecreasing. It is an elementary fact (cf. [5, Theorem 1]) that there are points $a = x_{-m} < \cdots < x_{-1} < x_0 = s$ and $t_{-m+1}, \ldots, t_0$ with $x_{k-1} \leqslant t_k \leqslant x_k$ and $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = -m+1, -m+2, \ldots, 0$. Hence

$$\left| \left(\int_{s}^{b} f \right) - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| \leqslant \varepsilon \tag{4}$$

by Henstock's Lemma, which proves that f is Riemann integrable on $[s, b]$. To show $\lim_{s \to a+} \int_s^b f = I$, let again $\varepsilon > 0$ and let $a < s < \min(a + \delta_{\varepsilon/2}(a), b)$. Choose $a = x_0 < s = x_1 < \cdots < x_n = b$ with $\max_{2 \leqslant k \leqslant n}(x_k - x_{k-1}) < \delta_{\varepsilon/2}(s)$ and set $t_k = x_{k-1}$, $k = 1,..., n$. Then

$$\left| I - \int_s^b f \right| \leqslant \left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| + \left| \sum_{k=2}^{n} f(t_k)(x_k - x_{k-1}) - \int_s^b f \right| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$$

by (3) and (4). This completes the proof of the theorem.

We now record some observations.

(a) Improper Riemann integrability of a real function f on $(a, b]$ $(-\infty < a < b < \infty, f(a) = 0)$ implies that for every $\varepsilon > 0$, $\delta_\varepsilon(t)$ of Theorem 2 can be taken also to be continuous on $(a, b]$. In fact, given $\varepsilon > 0$, refer to the proof of Theorem 2. In the x, y plane consider the infinite polygonal line $P_1 P_2 P_3 \cdots$ together with the point $(a, S - a)$, where $P_m = (c_{m-1}, \gamma_m)$, $m = 1, 2,...$. It is the graph of such a continuous function $y = \delta_\varepsilon(x)$.

(b) If f is a real function, improperly Riemann integrable on $[a, b)$ $(-\infty < a < b < \infty)$, the analogue of Theorem 2 is valid with $\delta_\varepsilon(t)$ a *nonincreasing* function on $[a, b)$.

(c) Analogues of Theorem 2 hold for improperly Riemann integrable functions on infinite intervals $(-\infty, b]$ or $[a, \infty)$. We state without proof the result for $[a, \infty)$.

THEOREM 3. *Let f be a real function on $[a, \infty)$, where $-\infty < a < \infty$. Then f is improperly Riemann integrable on $[a, \infty)$ (i.e., f is Riemann integrable on each $[a, b]$, $a < b < \infty$, and $\lim_{b \to \infty} \int_a^b f$ exists (finite)) iff there is a real number I with the following property: For each $\varepsilon > 0$ there is a positive nonincreasing function $\delta_\varepsilon(t)$ on $[a, \infty)$ and a number $B(\varepsilon) > a$ such that*

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \varepsilon$$

whenever $a = x_0 < \cdots < x_n$, $x_n > B(\varepsilon)$ and $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\varepsilon(t_k)$ for $k = 1,..., n$.

(d) Theorems 1 and 2 imply the (known) result [10, Theorem 1] that a dominantly integrable function is improperly Riemann integrable on $(0, 1]$. Similarly, Section 3 and Theorem 3 imply the (known) result [1, p. 931; 2,

p. 6] that a simply integrable function is improperly Riemann integrable on $[0, \infty)$.

(e) By using interval-valued functions $\delta(t)$, a definition of generalized Riemann integrability can be given which applies to both finite and infinite limits of integration; cf. [7, pp. 18, 23]. It is straightforward to show the equivalence of that definition to Definitions 1 and 3 above and to phrase the results of the present paper in terms of such interval-valued functions.

(f) McShane [8, 9] has studied similar Riemann-type integrals.

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